A General Approach for the Development of Unsplit-Field Time-Domain Implementations of Perfectly Matched Layers for FDTD Grid Truncation

Li Zhao and Andreas C. Cangellaris

Abstract—**It is shown that the anisotropic perfectly matched medium, proposed recently for the construction of reflectionless absorbing boundaries for differential equation-based electromagnetic simulations in unbounded domains, can be made equivalent to the Chew-Weedon perfectly matched medium developed from a modified Maxwell's system with coordinate stretching. Consequently, despite the apparently nonphysical coordinate stretching, Chew-Weedon's formulation, with an appropriate definition of the involved electric and magnetic fields, is merely an alternative mathematical form of Maxwell's system in an anisotropic medium. Finally, a more convenient time-domain implementation of the perfectly matched layer without splitting of the field components is derived.**

I. Introduction

OVER the past two years, Berenger's perfectly matched layer (PML) for the reflectionless truncation of differential equation-based wave simulations [1] has become the focus of extensive research [2]–[8]. This letter focuses on two specific approaches to the development of a PML. The first one was proposed by Chew and Weedon [2] and is closely related to the work by Rappaport [5]. The second one was proposed by Sacks *et al.* [7] and is based on a properly constructed anisotropic medium.

The approach proposed by Sacks *et al.* appears more attractive in view of the fact that there is no need for the Chew-Weedon modification of the spatial derivatives operators via coordinate stretching, and thus Maxwell's equations maintain their familiar physical form (except for the strange properties of the anisotropic medium). However, as shown below, these two approaches are mathematically identical, provided that the electric and magnetic fields present in the Chew-Weedon stretched-coordinate formulation are properly defined.

We prove this equivalence by introducing an alternative way of establishing the reflectionless properties of the anisotropic layer of Sacks *et al.* The advantage of this new formulation is that it leads directly to a new way of implementing the perfectly matched absorber in the time domain, without the need for splitting of the field components. Numerical results are used to demonstrate the validity and accuracy of this new implementation.

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The authors are with the Electromagnetics Laboratory, Department of Electrical and Computer Engineering, University of Arizona, Tucson, AZ 85721 USA.
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II. The Equivalence of the Two Approaches

With the assumption of a time dependence of the form $e^{j\omega t}$, Maxwell's equations in an anisotropic medium have the form

$$\nabla \times \mathbf{E} = -j\omega\bar{\mu} \cdot \mathbf{H} \tag{1a}$$
$$\nabla \times \mathbf{H} = j\omega\bar{\epsilon} \cdot \mathbf{E} \tag{1b}$$
$$\nabla \cdot (\bar{\epsilon} \cdot \mathbf{E}) = 0 \tag{1c}$$
$$\nabla \cdot (\bar{\mu} \cdot \mathbf{H}) = 0 \tag{1d}$$

where $\bar{\mu}$ and $\bar{\epsilon}$ are, respectively, the permeability and permittivity tensors of the medium. Following [7], we choose

$$\bar{\epsilon} = \epsilon(\mathrm{diag}\{a,b,c\}) = \epsilon[\Lambda] \tag{2a}$$
$$\bar{\mu} = \mu(\mathrm{diag}\{a,b,c\}) = \mu[\Lambda] \tag{2b}$$

where the elements of the diagonal matrix

$$[\Lambda] = \mathrm{diag}\{a,b,c\} \tag{3}$$

are, in general, complex, dimensionless, constants.

Let us define the field quantities $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ as follows:

$$\{\hat{E}_x, \hat{E}_y, \hat{E}_z\}^T = [G]^{-1}\{E_x, E_y, E_z\}^T \tag{4a}$$
$$\{\hat{H}_x, \hat{H}_y, \hat{H}_z\}^T = [G]^{-1}\{H_x, H_y, H_z\}^T \tag{4b}$$

where T denotes matrix tranposition and

$$[G] = \mathrm{diag}\{g_x, g_y, g_z\} \tag{5}$$

where g_x, g_y, g_z are, in general, complex constants. Their values will be defined later. Using the notation $\bar{\mathbf{G}}$ and $\bar{\mathbf{\Lambda}}$ to denote the tensors with matrix representations $[G]$ and $[\Lambda]$, respectively, Maxwell's equations can be written in terms of the *scaled* fields $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$

$$\nabla\times(\bar{\mathbf{G}} \cdot \hat{\mathbf{E}}) = -j\omega\mu\bar{\mathbf{\Lambda}} \cdot \bar{\mathbf{G}} \cdot \hat{\mathbf{H}} \tag{6a}$$
$$\nabla\times(\bar{\mathbf{G}} \cdot \hat{\mathbf{H}}) = j\omega\epsilon\bar{\mathbf{\Lambda}} \cdot \bar{\mathbf{G}} \cdot \hat{\mathbf{E}} \tag{6b}$$
$$\nabla \cdot (\epsilon\bar{\mathbf{\Lambda}} \cdot \bar{\mathbf{G}} \cdot \hat{\mathbf{E}}) = 0 \tag{6c}$$
$$\nabla \cdot (\mu\bar{\mathbf{\Lambda}} \cdot \bar{\mathbf{G}} \cdot \hat{\mathbf{H}}) = 0. \tag{6d}$$

Using the matrix representation of the curl operator

$$[\nabla\times] = \begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} \tag{7}$$

and choosing g_x, g_y and g_z such that

$$\left(\frac{g_x}{g_y}\right)^2 = \frac{b}{a}, \quad \left(\frac{g_y}{g_z}\right)^2 = \frac{c}{b}, \quad \left(\frac{g_z}{g_x}\right)^2 = \frac{a}{c} \tag{8}$$

it is a matter of straightforward matrix algebra to show that Maxwell's equations become

$$\nabla_{\mathbf{a}} \times \hat{\mathbf{E}} = -j\omega\mu\hat{\mathbf{H}} \tag{9a}$$
$$\nabla_{\mathbf{a}} \times \hat{\mathbf{H}} = j\omega\epsilon\hat{\mathbf{E}} \tag{9b}$$
$$\nabla_{\mathbf{a}} \cdot (\epsilon\hat{\mathbf{E}}) = 0 \tag{9c}$$
$$\nabla_{\mathbf{a}} \cdot (\mu\hat{\mathbf{H}}) = 0 \tag{9d}$$

where

$$\nabla_{\mathbf{a}} \stackrel{\text{def}}{=} \hat{\mathbf{x}}\frac{1}{\sqrt{bc}}\partial_x + \hat{\mathbf{y}}\frac{1}{\sqrt{ca}}\partial_y + \hat{\mathbf{z}}\frac{1}{\sqrt{ab}}\partial_z. \tag{10}$$

Clearly, using the notation

$$s_x = \sqrt{bc}, \quad s_y = \sqrt{ca}, \quad s_z = \sqrt{ab} \tag{11}$$

the system in (9) is mathematically equivalent to Chew-Weedon's stretched-coordinate formulation [2], provided that the fields in Chew-Weedon's formulation are scaled forms of the physical fields, defined by (4). Finally, we notice that the selection $g_x = \sqrt{bc}$, $g_y = \sqrt{ca}$, and $g_z = \sqrt{ab}$ satifies (8); hence, we have

$$g_x = s_x, \quad g_y = s_y \quad g_z = s_z. \tag{12}$$

III. The Reflectionless Interface

Clearly, the selection of the values of a, b, c (or, equivalently, the values of s_x, s_y, s_z) in order to effect a reflectionless planar interface between a medium $(\epsilon_1[\Lambda_1], \mu_1[\Lambda_1])$ and another medium $(\epsilon_2[\Lambda_2], \mu_2[\Lambda_2])$ can be based on a mathematical development identical to that of Chew and Weedon [2]. Therefore, the details of the development will not be presented here. We only mention that one must be cautious with the application of the boundary conditions at the interface between the two media. These conditions should be for the tangential components of the *physical* fields $\mathbf{E}, \mathbf{H}$, not the *scaled* fields $\hat{\mathbf{E}}, \hat{\mathbf{H}}$.

As shown in [2], plane wave solutions of (9) are characterized by the dispersion relation

$$\omega^2\mu\epsilon = (k_x/s_x)^2 + (k_y/s_y)^2 + (k_z/s_z)^2 \tag{13}$$

which is satisfied by

$$k_x = ks_x \sin\theta\cos\phi \tag{14a}$$
$$k_y = ks_y \sin\theta\sin\phi \tag{14b}$$
$$k_z = ks_z \cos\theta \tag{14c}$$

where $k = \omega\sqrt{\mu\epsilon}$. Furthermore, the wave impedance, η, is not affected by the coordinate stretching and is given by the familiar expression $\eta = \sqrt{\mu/\epsilon}$.

Let us consider the planar interface between two media. We assume that the interface is parallel to the x-y plane in a cartesian coordinate system. The fields in medium one satisfy (9) with material properties $\epsilon_1[\Lambda_1]$, $\mu_1[\Lambda_1]$, and corresponding stretching parameters s_{x1}, s_{y1}, s_{z1}. The fields in medium two satisfy (9) with material properties $\epsilon_2[\Lambda_2]$, $\mu_2[\Lambda_2]$, and corresponding stretching parameters s_{x2}, s_{y2}, s_{z2}. Following the development in [2], it can be shown that the interface can be rendered reflectionless for all frequencies and all angles of incidence of a plane wave propagating, say, from medium one to medium two, if

$$\epsilon_1 = \epsilon_2, \quad \mu_1 = \mu_2, \tag{15a}$$
$$s_{x1} = s_{x2}, \quad s_{y1} = s_{y2}. \tag{15b}$$

Furthermore, in view of (14c), attenuation of the transmitted wave in medium two can be effected by proper selection of s_{z2}. Thus, a reflectionless (perfectly matched) medium is constructed.

For example, consider the case where medium one is homogeneous and isotropic. Then $[\Lambda_1]$ is the identity matrix, and thus $s_{x1} = s_{y1} = s_{z1} = 1$. From (15), the interface will be reflectionless if (14a) is satisfied and the elements of $[\Lambda_2]$ are such that $s_{x2} = s_{y2} = 1$, This, in view of (11), results in $a_2 = b_2$ and $c_2a_2 = 1$ which, in turn, give $s_{z2} = a_2$. Selecting a_2 to be complex results in attenuation of the wave as it propagates in the anisotropic, perfectly matched medium.

If we let $a_2 = 1 + (\sigma/j\omega\epsilon)$ (where we have set $\epsilon_1 = \epsilon_2 = \epsilon$, $\mu_1 = \mu_2 = \mu$), Maxwell's first curl equation inside the anisotropic perfectly matched medium becomes

$$\frac{1}{\mu}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = -j\omega H_x - \frac{\sigma}{\epsilon}H_x \tag{16a}$$
$$\frac{1}{\mu}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = -j\omega H_y - \frac{\sigma}{\epsilon}H_y \tag{16b}$$
$$\left(1 + \frac{\sigma}{j\omega\epsilon}\right)\frac{1}{\mu}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -j\omega H_z. \tag{16c}$$

With regards to the time-dependent form of the above equations, we observe that (16a) and (16b) have the standard form for wave propagation in a lossy medium with magnetic conductivity $\sigma^* = \sigma(\mu/\epsilon)$. However, (16c), the equation for the component normal to the interface between the homogeneous medium and the perfectly matched, anisotropic absorber requires special attention. Transforming (16c) to the time domain we obtain

$$\frac{1}{\mu}\left(\frac{\partial E_y(t)}{\partial x} - \frac{\partial E_x(t)}{\partial y}\right) = -\frac{\partial H_z(t)}{\partial t} - \frac{\sigma}{\mu\epsilon}\int_0^t \left(\frac{\partial E_y(\tau)}{\partial x} - \frac{\partial E_x(\tau)}{\partial y}\right)d\tau. \tag{17}$$

The integral on the right-hand side of (17) is simply the time integration of the z component of $\nabla \times \mathbf{E}$. It is interpreted as a time-dependent source present only within the perfectly matched, anisotropic absorber. The use of time-dependent source terms for the time-dependent implementation of the unsplit field formulation of Berenger's perfectly matched layer [6], has also been discussed by Veihl and Mittra [8]. Notice that in the FDTD approximation of this equation, the value of H_z at time $t = (n + 1/2)\Delta t$ is found from its value at $t = (n - 1/2)\Delta t$, the value of the z component of $\nabla \times \mathbf{E}$ calculated at $t = n\Delta t$ and the time integral of this term calculated up to $t = n\Delta t$.

From duality it is apparent that a system similar to (16) is obtained from Maxwell's curl equation for the magnetic field. Thus, a time-dependent source term, involving the time integral of the z component of $\nabla \times \mathbf{H}$, appears in the update equation for E_z (the component of $\mathbf{E}$ normal to the interface). Thus, for perfectly matched, anisotropic media with one direction of attenuation, two time-dependent sources appear in the time-dependent form of Maxwell's equations.

From (9), it can be shown that the above result can be extended to perfectly matched, anisotropic media with more than one directions of attenuation.

IV. Numerical Validation

In order to validate numerically the derived time-dependent source implementation of the anisotropic, perfectly matched medium, a numerical experiment was attempted in two dimensions. A point source at the center of a 100 × 50-cell domain, Ω_N, on the x-y plane was excited by a smooth compact pulse. The polarization of choice was transverse magnetic (TM); thus, the field components H_x, H_y, E_z were involved. The domain of computation was terminated by either Berenger's PML backed by perfect electric conductors, or by an anisotropic PML, effected using (9), also backed by perfect electric conductors. The benchmark FDTD solution, with zero truncation boundary reflections, was obtained by simulating radiation by the aforementioned point source in a much larger domain, Ω_L, centered at the point source discretized by a finite-difference grid of same cell size as that for Ω_N and with truncation boundaries placed sufficiently far away to provide for causal isolation for all points in Ω_N over the time interval used for the comparisons.

The error due to numerical reflections caused by the presence of the conductor-backed PML's was obtained by subtracting at each time step the field at any grid point inside Ω_N from the field at the corresponding point in Ω_L. Fig. 1 compares the global error energy (sum of the squares of the error at each grid point in Ω_N) versus time for the standard Berenger's PML to that for the proposed time-dependent source implementation of the anisotropic PML. Two cases were considered. One with a four-element PML and one with an eight-element PML. For both cases, a quadratic variation in PML conductivities was assumed, with maximum value chosen for theoretical reflection coefficient of 10^{-5} at normal incidence. Clearly, for the case of the eight-element PML, both methods are as effective in keeping the reflection error close to its theoretically predicted value.

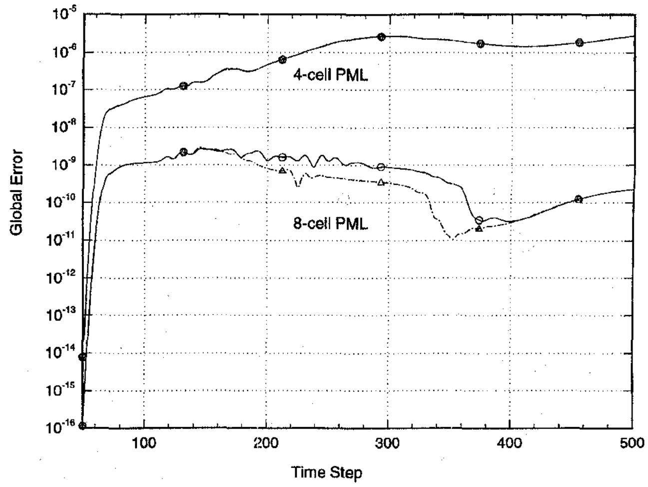

Fig. 1. Global error energy within a 100 × 50-cell FDTDgrid with a pulsed TM point source at its center. Grid truncation was effected using Berenger's split-field PML, as well as the proposed unsplit-field implementation of an anisotropic PML. Both methods produce the same error for the case of a four-layer PML. For the case of an eight-layer PML, Berenger's PML (triangles) has lower error at earlier times; however, both methods converge to the same global error at late times.

V. Conclusion

In conclusion, a mathematical formulation has been presented for an anisotropic, perfectly matched medium that can be used for numerical grid truncation in both frequency- and time-dependent wave simulations using FDTD techniques. The proposed formulation helps prove the equivalence of the Chew-Weedon stretched-coordinate formulation, modified by appropriate field scaling, with the anisotropic perfectly matched medium proposed first by Sacks *et al.* The proposed formulation has the advantage that it can be implemented in the time domain without any splitting of the fields.

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